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POISSON BRACKETS AND CLEBSCH REPRESENTATIONS FOR MAGNETOHYDRODYNAMICS, MULTIFLUID PLASMAS, AND ELASTICITY

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Poisson brackets are constructed by the same mathematical procedure for three physical theories: ideal magnetohydrodynamics, multifluid plasmas, and elasticity. Each of these brackets is given a simple Lie-algebraic interpretation. Moreover, each bracket is induced to physical space by use of a canonical Poisson bracket in the space of Clebsch potentials, which are constructed for each physical theory by the standard procedure of constrained Lagrangians.

1. Introduction

Since Gardner's revelation [1] (see also Arnol'd [20]) of the noncanonical Poisson structure for the Korteweg–de Vries equation of weakly nonlinear dispersive waves, and subsequent discovery by Gel'fand and Dikii of the Hamiltonian form of the general theory of scalar Lax equations (see, e.g. [2], ch. I), the Hamiltonian formalism has surfaced in a number of hydrodynamically-minded physical models: two-dimensional shallow water waves [3], ideal compressible hydrodynamics and a variety of condensed matter problems [4], ideal magnetohydrodynamics [5, 6] multifluid plasmas [7], etc. Indeed with so much current activity in this area, one would expect prompt disposal of the problems which remain still Poisson-bracketless.

Thus, the center of attention in questions related to Hamiltonian formalism in classical field theories moves from the task of simply uncovering the relevant Poisson brackets, toward general inquiry about the nature of these brackets and the procedures leading to their unmasking. It is on these two topics that we concentrate in this paper, although the "final solution" awaits a more extensive treatment. By "nature" we mean here the correspondence of these brackets to Lie algebras. Another point of view is developed in [16] and [17], where the nature of those brackets is studied group theoretically.

As the title promises, we treat three physical models, for simplicity working in $\mathbb{R}^n$. In each of these cases, we proceed via the same four steps:

a) We enlarge the original, physical system by introducing "parasite" variables – the so-called Clebsch potentials* – together with a variational principle for the dynamical equations in the enlarged Clebsch space;

b) Next, we show that dynamics in the Clebsch space is governed by Hamiltonian mechanics, in standard form, with a Hamiltonian which is the energy of the original system, but expressed in the Clebsch space;

c) We then determine whether the canonical Poisson structure that lives in the Clebsch space can be induced properly into the space of physical variables alone;

* The history of Clebsch representations can be gleaned from, e.g. the reviews [8] and [14].

d) Finally, we pinpoint the relevant Lie algebra which is responsible for the Poisson bracket in the physical space.

The mathematical notions necessary for the steps c) and d), are described in sections 3 and 4 respectively. Magnetohydrodynamics (MHD) which is treated in a relatively ad hoc manner in two previous papers [5, 6] is analyzed in considerable detail in sections 1 and 2 and also in sections 3, formula (24), and section 4, formulae (41)–(45) and (51)–(55). Multifluid plasmas are discussed in section 5, where one of the two derived Poisson brackets turns out to be exactly the bracket of Spencer and Kaufman [7]. Finally, elasticity theory is treated in section 6. We conclude this introduction by mentioning the physically meaningless, but mathematically amusing corollary of one of the last formulae (84) in the elasticity analysis: In one dimension, the Poisson bracket of MHD in the space of magnetic potentials is exactly the same as the Poisson bracket for elasticity in the space of Lagrangian deformations.

2. Magnetohydrodynamics (MHD)

The MHD fluid has mass density, ρ, and specific entropy, η. It moves through Euclidean space $\mathbb{R}^n$ with positions x_i and velocities v_j and carries an embedded magnetic field, B_{ij}, expressible in terms of a vector potential, A_j, according to $B_{ij} = A_{i,j} - A_{j,i}$ with subscript notation also for partial derivatives.

In terms of momentum density $M_j = \rho v_j$ the MHD equations are

$$\dot{M}_i = -[M_iM_j/\rho + \delta_{ij}(p - \tfrac{1}{4}\operatorname{Tr} B^2) - B_{ik}B_{kj}]_{,j}, \tag{1}$$

$$\dot{\rho} = -M_{j,j}, \tag{2}$$

$$\dot{\eta} = -\frac{M_j}{\rho}\,\eta_{,j}, \tag{3}$$

$$\dot{B}_{ij} = (B_{jk}M_k/\rho)_{,i} - (B_{ik}M_k/\rho)_{,j}, \tag{4}$$

where superscript $\cdot$ denotes partial time derivative $\partial/\partial t$, and we sum on repeated indices. Eq. (1) is the hydrodynamic motion equation expressed in conservative form as the divergence of the stress tensor for MHD, where $\operatorname{Tr} B^2 = B_{ij}B_{ji}$ in the stress tensor. The fluid pressure p is determined as a function of ρ and η from a prescribed relation for the specific internal energy, $e(\rho, \eta)$ combined with the first law of thermodynamics,

$$\mathrm{d}e = e_\rho\,\mathrm{d}\rho + e_\eta\,\mathrm{d}\eta = \rho^{-2}p\,\mathrm{d}\rho + T\,\mathrm{d}\eta, \tag{5}$$

where T is temperature.

In what follows, we shall work in terms of the vector potential $\mathbf{A}$, although we shall also comment on counterparts of the results in terms of the magnetic fields.

Faraday's law, eq. (4) for MHD, follows from an evolution equation for vector potential A_i,

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} := \dot{A}_i + v_kA_{i,k} = -A_kv_{k,i}, \tag{6}$$

which implies the relation

$$A_i = A_k^0\frac{\partial X_k^0}{\partial x_i} \tag{7}$$

where $A_k^0(\boldsymbol{x}, t)$ and $X_k^0(\boldsymbol{x}, t)$ convect with the fluid,

$$\frac{\mathrm{d}A_k^0}{\mathrm{d}t} = 0 = \frac{\mathrm{d}X_k^0}{\mathrm{d}t}. \tag{8}$$

As the notation suggests, $X_k^0(\boldsymbol{x}, t)$ can be regarded as the Lagrangian coordinate: the initial position of the fluid particle that occupies position $\boldsymbol{x}$ at time t, and, by (7), $A_k^0(\boldsymbol{x}, 0)$ is the initial spatial distribution of magnetic vector potentials.

The MHD system can be expressed as a Hamiltonian system $\dot{F} = \{H, F\}$ with Hamiltonian density

$$H = \frac{M^2}{2\rho} + \rho e(\rho, \eta) - \tfrac{1}{4}\operatorname{Tr} B^2, \tag{9}$$

where $\operatorname{Tr} B^2 = (A_{i,j} - A_{j,i})(A_{j,i} - A_{i,j})$. The Poisson bracket $\{F, G\}$ for densities F and G is defined to be [6]

$$\begin{aligned}-\{F, G\} = {} & \frac{\delta G}{\delta\rho}\,\partial_j\rho\,\frac{\delta F}{\delta M_j} + \frac{\delta G}{\delta\sigma}\,\partial_j\sigma\,\frac{\delta F}{\delta M_j} \\ & + \frac{\delta G}{\delta M_j}\left[\rho\partial_j\,\frac{\delta F}{\delta\rho} + \sigma\partial_j\,\frac{\delta F}{\delta\sigma} + (M_k\partial_j + \partial_k M_j)\,\frac{\delta F}{\delta M_k} + (A_{j,k} - A_{k,j} + A_j\partial_k)\,\frac{\delta F}{\delta A_k}\right] \\ & + \frac{\delta G}{\delta A_j}(A_{j,k} - A_{k,j} + \partial_j A_k)\,\frac{\delta F}{\delta M_k},\end{aligned} \tag{10}$$

where

$$\sigma = \rho\eta \tag{10'}$$

is entropy per unit volume.

The MHD equations are then identical to

$$\dot{F} = \{H, F\}, \qquad F \in \{\rho, \sigma, M_i, A_i\} \tag{11}$$

for Hamiltonian H given by eq. (9) and bracket given by eq. (10). The bracket (10) for MHD, as well as its counterparts for multifluid plasmas and elasticity will be constructed here from a constrained Lagrangian procedure. Moreover, the interpretation of these brackets will be given in terms of differential Lie algebras.

2. Clebsch representation for MHD

The Poisson bracket (10) may be constructed by restriction of another, canonical, bracket which derives from a constrained Hamilton's principle.

There are numerous variational formulations of fluid dynamics (see, e.g. [14]). For example, extremals of the following variational principle

$$\delta \int \mathrm{d}t\, \mathrm{d}^n x \mathscr{L} = 0 \tag{12}$$

yield MHD flows when the Lagrangian density is given by

$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}\rho v^2 - \rho e(\rho, \eta) + \tfrac{1}{4}\operatorname{Tr} B^2 + \phi(\dot\rho + \operatorname{div}\rho\boldsymbol{v}) - \beta(\dot\eta + \boldsymbol{v}\cdot\nabla\eta) \\ & - \gamma_k(\dot{X}_k^0 + \boldsymbol{v}\cdot\nabla X_k^0) - f_k(\dot{A}_k^0 + \boldsymbol{v}\cdot\nabla A_k^0).\end{aligned} \tag{13}$$

Introduction in (13) of functions $\{\phi, \beta, \gamma_k, f_k\}$ as Lagrange multipliers has imposed the MHD subsidiary equations (2), (3), and (8) upon the extremals of L.

Separate variations in Hamilton's principle (12) with respect to $\boldsymbol{v}$, ρ, η, X_k^0, A_k^0 produce the following auxiliary equations:

$$\delta \boldsymbol{v}:\quad \rho \boldsymbol{v} = \boldsymbol{M} = \rho \nabla \phi + \beta \nabla \eta + \gamma_k \nabla X_k^0 + f_k \nabla A_k^0, \tag{14a}$$

$$\delta \rho:\quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{v^2}{2} - (e + p/\rho), \tag{14b}$$

$$\delta \eta:\quad \frac{\mathrm{d}}{\mathrm{d}t}(\beta/\rho) = \frac{\partial e(\rho, \eta)}{\partial \eta} = T, \tag{14c}$$

$$\delta X_k^0:\quad \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_k/\rho) = 0, \tag{14d}$$

$$\delta A_k^0:\quad \frac{\mathrm{d}}{\mathrm{d}t}(f_k/\rho) = -\frac{1}{\rho} X_{k,l}^0 B_{lm,m}. \tag{14e}$$

Eqs. (14b) through (14e) together with the constraint equations (2), (3), and (8) imposed by Lagrange multipliers can be rewritten as canonical equations for a Hamiltonian system,

$$\dot{p}_\alpha = -\{p_\alpha, H\} = -\frac{\delta H}{\delta q^\alpha}, \quad \dot{q}^\alpha = -\{q^\alpha, H\} = \frac{\delta H}{\delta p_\alpha}, \tag{15}$$

with Hamiltonian density

$$H = \tfrac{1}{2}\rho v^2 + \rho e(\rho, \eta) - \tfrac{1}{4}\mathrm{Tr}\, B^2 \tag{16}$$

and canonical variables given by

$$q^\alpha \in (\rho, \beta, \gamma_k, f_k), \quad p_\alpha \in (\phi, \eta, X_k^0, A_k^0). \tag{17}$$

Then, from canonical equations (15) the equation of motion (1) may be recovered by algebraic manipulation and use of relation (14a). Relations of the type (14a) in the form $\boldsymbol{M} = p_\alpha \nabla q_\alpha$ with canonically conjugate p_α and q_α have been known in the literature (see, e.g., the reviews [8] and [14]) as Clebsch representations.

In three space dimensions, another, different, Clebsch representation and Hamiltonian formulation for MHD, originally due to Zakharov and Kuznetsov [9], can be obtained by imposition of Faraday's law (4) as a variational constraint directly in terms of magnetic field, instead of in the potential form (8). The Clebsch representation so obtained involves an auxiliary variable, a Lagrange multiplier which is canonically conjugate to the magnetic field. Unfortunately though, subsequent restriction of the canonical bracket to the physical variables $\{\rho, \sigma, M_i, B_{ij}\}$ does not eliminate all of the auxiliary Clebsch potentials from the resulting bracket. Still another Clebsch representation exists in the special case of three space dimensions [8]. This representation does properly restrict to a correct Poisson bracket for functionals of the magnetic field, $\boldsymbol{B}$ (as opposed to functionals of the vector potential, $\boldsymbol{A}$). As these examples indicate, there may exist many different Clebsch representations for the same problem. Apparently, at least some of them have a Lie-algebraic meaning (see, e.g. [15], ch. VIII, sec. 4, or [17]). But it remains an open problem to understand why multiple Clebsch representations exist, in cases where a given physical system is not accompanied by its Lie algebraic interpretation.

We shall soon see that restriction of the canonical bracket (15) to physical variables $\{\rho, \sigma, M_i, A_i\}$ through the Clebsch map (14a) produces the Poisson bracket (10). But first, in the next section, we discuss the induction procedure for general Poisson brackets.

3. The induction procedure

We describe here how to transform Poisson brackets from one space to another. Suppose we have a space Z with dependent variables $\{Z^\mu\}$, independent variables $(x_1, \ldots, x_n)$ being fixed once and for all. Assume also that we have a Poisson bracket structure in the Z-space, sometimes also called a Hamiltonian structure. This structure consists of a skew-adjoint matrix differential operator $B = (B^{\mu\nu})$ which to any Hamiltonian H on Z assigns an evolutionary vector field $X_H = B(\delta H/\delta Z)$ which has the following equations of trajectories:

$$\frac{\partial Z^\mu}{\partial t} = B^{\mu\nu} \frac{\delta H}{\delta Z^\nu}. \tag{18}$$

The Poisson bracket of two Hamiltonians H and F is defined as

$$\{H, F\} = X_H(F) \equiv \frac{\delta F}{\delta Z^\mu} B^{\mu\nu} \frac{\delta H}{\delta Z^\nu}, \tag{19}$$

where $\equiv$ means: equality modulo total derivatives (divergences).

Not every skew-adjoint B implies satisfaction of the Jacobi identity for the Poisson bracket. To determine whether a given bracket satisfies the Jacobi identity is usually a tedious job (see, e.g. ch. I of [2] for an extensive discussion of Hamiltonian structures). However, there are some situations where a proper matrix B can be constructed very simply. One such situation is connected with representations of Lie algebras and will be discussed in the next section.

Here we describe a procedure of induction which produces new Hamiltonian operators B from old ones. Suppose we have another space $V = \{v^\alpha\}$ in addition to Z, with a map between them $\Delta: Z \to V$ given by a set of (nonlinear) differential operators

$$v^\alpha = v^\alpha(x_j, Z^\mu_{(\sigma)}), \tag{20}$$

where

$$Z^\mu_{(\sigma)} = \frac{\partial^\sigma Z^\mu}{\partial x^\sigma}, \quad \partial^\sigma = \frac{\partial^\sigma}{\partial x^\sigma} = \left(\frac{\partial}{\partial x_1}\right)^{\sigma_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\sigma_n}, \quad \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{Z}^n_+.$$

Let $J = (J^{\alpha\mu})$ be the Frechet derivative of the map Δ, i.e. the matrix differential operator

$$J^{\alpha\mu} = \frac{\partial v^\alpha}{\partial Z^\mu_{(\sigma)}} \partial^\sigma, \tag{21}$$

and let $J^\dagger$ be the adjoint of J. Consider the new matrix $\bar{B} = (\bar{B}^{\alpha\beta})$:

$$\bar{B} = JBJ^\dagger. \tag{22}$$

If coefficients of $\bar{B}$ can be expressed in terms of variables $(x, v^\alpha_{(\sigma)})$ only, the matrix $\bar{B}$ is very likely to define a Hamiltonian structure in V-space which is compatible, via the map Δ, with the structure defined by the matrix B and Z-space. We hasten to add, however, that there is no general guarantee that the matrix $\bar{B}$ will indeed define a Hamiltonian structure in V-space. To see why, let us consider first the simplest case of classical mechanics, when $n = 0$ and Z and V are both finite dimensional.

Then $\bar{B}$ defines a Hamiltonian structure only on the Image of Δ, and this can be only part of V. If, however, Δ is an epimorphism, then $\bar{B}$ provides a Hamiltonian structure on the whole manifold V. Now, if we are not in the situation of classical mechanical type, that is, when $n > 0$, Z and V are infinite-dimensional jet-spaces (geometric or algebraic), and when Δ is indeed a differential operator, then the problem arises of whether Δ is an epimorphism, or, equivalently, if the map Δ^* on functions is injective. Even in simplest situations, this problem is quite tedious to untangle (see, e.g. [10], for the case when Δ is a differential version of the map given in algebra by the elementary symmetric functions). Our approach to the whole question is as follows. We compute $\bar{B}$ and make sure that its coefficients live on V; to do this one just multiplies matrices in (22). We spare the reader all intermediate matrix scenes and simply give the resulting bracket as in (19). We then independently verify that our matrix $\bar{B}$ does indeed define a Hamiltonian structure; since all our matrices $\bar{B}$ have Lie algebraic origin, this is done easily (see section 4 for an outline of the theory).

In the MHD case, B is canonical in the space with coordinates p_α, q_α (formerly Z^μ)

$$\{F, G\} = \frac{\delta F}{\delta p_\alpha}\frac{\delta G}{\delta q^\alpha} - \frac{\delta G}{\delta p_\alpha}\frac{\delta F}{\delta q^\alpha}. \tag{23}$$

For MHD, Δ is given by (10′), (14a), (7) as

$$\rho = p_1, \quad \sigma = p_1 q^2, \qquad M_i = p_\alpha q^\alpha_{,i}, \quad A_i = \sum_{\substack{\beta\in I\\ \gamma\in J}} q^\beta q^\gamma_{,i}, \tag{24}$$

where two disjoint subsets of indices are such that $J, I \in \{1, 2\}$ and $|J| = |I|$ (the number of elements is the same in both J and I). The matrix $\bar{B}$ that results from (22) under map (24) provides the Poisson bracket for MHD: formula (10).

4. Poisson brackets associated with Lie algebras

In this section we recall briefly how Hamiltonian structures are generated by Lie algebras, and derive a few basic formulas for later use. For simplicity we work in coordinates, by fixing a basis in each space.

First, consider the finite-dimensional case. Let L be a Lie algebra over a field $\mathscr{k}$ (say, $\mathbb{R}$ or $\mathbb{C}$). Fix a basis $(e_1, \dots, e_m)$ in L and dual basis $(e_1^*, \dots, e_m^*)$ in the dual space L^* to L. Let C^k_{ij} be structure constants of L: if $X = X_i e_i$, $Y = Y_j e_j$, then

$$(X\Delta Y)_k = C^k_{ij} X_i Y_j, \tag{25}$$

where Δ denotes multiplication in L.

Let $u = (u_1, \dots, u_m)$ be coordinates in L^* and $\langle u, X\rangle = u_k X_k$ be the pairing between L^* and L. Define the skew matrix B on L^* by

$$\langle u, X\Delta Y\rangle = X_i B^{ij} Y_j, \qquad \forall X, Y \in L. \tag{26}$$

Thus,

$$B^{ij} = u_k C^k_{ij}. \tag{27}$$

As is well known (see, e.g. [11]), the matrix B is Hamiltonian, i.e. the corresponding Poisson bracket

$$\{F, G\} = \{F, G\}_{(L^*)} = \frac{\delta G}{\delta u_i} u_k C^k_{ij} \frac{\delta F}{\delta u_j} \tag{28}$$

does satisfy the Jacobi identity.

Let $\mathfrak{g}$ be another Lie algebra, $\phi\colon L \to \mathfrak{g}$ be a homomorphism of Lie algebras, $\phi^*\colon \mathfrak{g}^* \to L^*$ be the dual map. Denote by $\bar{\phi} = (\phi^*)^*\colon C^\infty(L^*) \to C^\infty(\mathfrak{g}^*)$ the induced map on functions. Naturally, ϕ is canonical, that is,

$$\bar{\phi}(\{F, G\}_{(L^*)}) = \{\bar{\phi}(F), \bar{\phi}(G)\}_{(\mathfrak{g}^*)}, \qquad \forall F, G \in C^\infty(L^*). \tag{29}$$

The formulae above for the finite-dimensional case are relevant to classical mechanics. In the field-theoretic situation which concerns us in this paper, one needs only to introduce minor changes, as follows. Let K be now a differential algebra (say, $C^\infty(\mathbb{R}^n)$) and L be a free module over K (say, K^m). Let $\partial_i (= \partial/\partial x_i$ for $K = C^\infty(\mathbb{R}^n))$, $i = 1, \ldots, n$, denote commuting derivations of K, and the same notation stand for their unique extensions on L. We let ∂^σ denote $\partial_1^{\sigma_1}, \ldots, \partial_n^{\sigma_n}$ for a multi-index $\sigma = (\sigma_1, \ldots, \sigma_n) \in Z^n_+$. Suppose that L is a differential algebra: if $X = X_i e_i$, $Y = Y_j e_j$ then

$$(X \Delta Y)_k = C^k_{ij,\sigma,\nu} \partial^\sigma X_i \partial^\nu Y_j, \qquad C^k_{ij,\sigma,\nu} \in K \tag{30}$$

with finite sum for every k, where $X \Delta Y$ denotes multiplication in L and $(e_1, \ldots, e_m)$ is a basis in L. We are interested in the case when L is a Lie algebra.

Let $u_1, \ldots, u_m$ be differentially independent variables, which can informally thought of as "coordinates on L^*". We write

$$\langle u, X \rangle = u_k X_k. \tag{31}$$

Let K_u denote the differential ring in variables $u_1, \ldots, u_m$ over K, in other words, K_u is a polynomial ring in variables $u_i^{(\sigma)}$ with derivations ∂_k extended to act on K_u as: $\partial_k(u_i^{(\sigma)}) = u_i^{(\sigma + 1_k)}$, $k = 1, \ldots, n$. This K_u can be thought of as "functions on L^*."

Finally, we let $\operatorname{Im} \mathscr{D}$ denote $\Sigma_k \operatorname{Im} \partial_k$ and we write $a \equiv b$ if $(a - b) \in \operatorname{Im} \mathscr{D}$.

We introduce the skew-adjoint matrix B in K_u by the formula

$$\langle u, X \Delta Y \rangle \equiv X_i B^{ij} Y_j, \qquad \forall X, Y \in L. \tag{32}$$

Thus,

$$B^{ij} = (-\partial)^\sigma u_k C^k_{ij,\sigma,\nu} \partial^\nu \tag{33}$$

and the corresponding Poisson bracket

$$\{F, G\} \equiv \frac{\delta G}{\delta u_i} (-\partial)^\sigma u_k C^k_{ij,\sigma,\nu} \partial^\nu \frac{\delta F}{\delta u_j} \tag{34}$$

does satisfy the Jacobi identity [12, 15]. If $H \in K_u$, then "trajectories" of the evolution field $X_H = B(\delta H/\delta u)$ satisfy

$$\dot{u}_i = B^{ij} \frac{\delta H}{\delta u_j}. \tag{35}$$

Let $\mathfrak{g}$ be another differential Lie algebra over K with the basis $\{\bar{e}_j\}$ and let K_v be the "ring of functions" on $\mathfrak{g}^*$. Let $\phi\colon L \to \mathfrak{g}$ be a homomorphism of Lie algebras, given by a linear differential operator ϕ: if $X = X_i e_i$ then $\phi(X) = R = Q_i \bar{e}_i$, where

$$Q_i = \phi_i(X) = \phi_{ij}(X_j), \qquad \phi_{ij} = b_{ij}^{\sigma}\partial^{\sigma}, \quad b_{ij}^{\sigma} \in K. \tag{36}$$

As in the finite-dimensional case, we let $\bar{\phi} = (\phi^*)^*$ denote a homomorphism $K_u \to K_v$ uniquely defined by its properties

$$1)\ \bar{\phi}\partial_l = \partial_l \bar{\phi}_1, \qquad l = 1, \ldots, n,$$

$$2)\ \bar{\phi}(u_j) = \phi_{ij}^{\dagger}(v_i). \tag{37}$$

Again, ϕ is canonical, i.e.

$$\phi(\{H, F\}) \equiv \{\phi(H), \phi(F)\}, \qquad \forall F, H \in K_u. \tag{38}$$

4.1. *Semidirect products*

We will use this construction in the following form. Let V_1 and V_2 be K-modules and R_i: $L \to \mathrm{Diff}(V_i)$ be representations of L by (linear) differential operators in V_i. Let Δ: $V_1 \to V_2$ be a linear differential operator for which actions R_1 and R_2 are compatible:

$$\Delta R_1(X) = R_2(X)\Delta, \qquad \forall X \in L. \tag{39}$$

Consider new Lie algebras $L \ominus V_i$ (semidirect products). Then operator Δ defines a Lie algebra homomorphism $\mathrm{id} \ominus \Delta$ (described above under notation ϕ) which we again denote by Δ:

$$\Delta: L \ominus V_1 \to L \ominus V_2, \tag{40}$$

given as

$$\Delta(X; v_1) = (X; \Delta(v_1)).$$

In our applications L will be always the Lie algebra of vector fields on $\mathbb{R}^n, \mathscr{D}(\mathbb{R}^n)$ the V_j will be direct sums of modules of differential forms Λ^i on which vector fields act naturally by Lie derivatives, and Δ will be direct sums of differentials d_i: $\Lambda^i \to \Lambda^{i+1}$.

To illustrate details, let us consider firstly the case of $L = \mathscr{D}(\mathbb{R}^n)$ itself. We identify $\mathscr{D}(\mathbb{R}^n)$ with K^n in the following way: $X = (X_1, \ldots, X_n)$ acts as $X_i(\partial/\partial x_i)$ on $\Lambda^*(\mathbb{R}^n)$. To conform with physical usage we denote by M_i the dual coordinates on L^* (previously denoted as u_i).

To compute the corresponding Poisson bracket, we use (32). Let $Y = Y_j(\partial/\partial x_j)$. Recall that multiplication in $\mathscr{D}(\mathbb{R}^n)$ is defined by the Lie bracket,

$$[X, Y] = (X_j Y_{i,j} - Y_j X_{i,j})\partial_i.$$

Then eq. (32) transcribes for this case into

$$M_i(X_j Y_{i,j} - Y_j X_{i,j}) \equiv X_j M_i Y_{i,j} + X_i (M_i Y_j)_{,j} = X_i [M_j \partial_i + \partial_j M_i] Y_j$$

and we get the skew-adjoint matrix

$$B^{ij} = M_j \partial_i + \partial_j M_i. \tag{41}$$

Consequently, the Poisson bracket (34) is realized as

$$\{F, G\} = \frac{\delta G}{\delta M_i}(M_j \partial_i + \partial_j M_i)\frac{\delta F}{\delta M_j} \equiv M_j \left(\frac{\delta G}{\delta M_i}\partial_i \frac{\delta F}{\delta M_j} - \frac{\delta F}{\delta M_i}\partial_i \frac{\delta G}{\delta M_j}\right). \tag{42}$$

We denote this bracket $\{F, G\}_M$: in the general $L \ominus V$ case $\{F, G\}_M$ adds to those pieces of the bracket which correspond to action of L on V.

We compute each piece of the bracket separately, for the action of L on $\Lambda^i = \Lambda^i(\mathbb{R}^n)$, with $i = 0, 1, n-1, n$. Let us start with $\Lambda^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. Let ρ denote the corresponding dual coordinate. If we write elements of $L \ominus \Lambda^0$ as pairs $(X; f)$ then, for another element $(Y; g) \in L \ominus \Lambda^0$, we have

$$[(X; f), (Y; g)] = ([X, Y]; X(g) - Y(f)).$$

Thus, if here and below we agree to omit $\langle M, [X, Y]\rangle$ terms already worked out above in (42), our formula (32) directs us to

$$\rho(X(g) - Y(f)) = \rho(X_i g_{,i} - Y_i f_{,i}) \equiv X_i(\rho\partial_i)g + f(\partial_i \rho)Y_i. \tag{43}$$

Thus, we get the piece of the Poisson bracket corresponding to Λ^0:

$$\{F, G\}_\rho = \frac{\delta G}{\delta M_i}\rho\partial_i \frac{\delta F}{\delta \rho} + \frac{\delta G}{\delta \rho}\partial_i \rho \frac{\delta F}{\delta M_i} \equiv \rho\left(\frac{\delta G}{\delta M_i}\partial_i \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta M_i}\partial_i \frac{\delta G}{\delta \rho}\right). \tag{44}$$

For a direct sum of several Λ^0's, we prescribe indices to corresponding ρ's: ρ_1, ρ_2, etc.

At this point we pause for a moment to see what we have obtained so far. Let us take

$$\{F, G\}_M + \{F, G\}_{\rho_1} + \{F, G\}_{\rho_2}. \tag{45}$$

Comparing (45) with (10) we see that this is exactly the Poisson bracket for compressible, ideal hydrodynamics with $\rho_1 = \rho$, $\rho_2 = \sigma$, and with magnetic terms absent.

Next let us take $V = \Lambda^n$. Let $\omega = f\,\mathrm{d}^n x$, $\nu = g\,\mathrm{d}^n x$ be arbitrary elements of Λ^n, where $f, g \in \Lambda^0$, $\mathrm{d}^n x = \mathrm{d}x_1 \cdots \mathrm{d}x_n$. Let θ denote the dual coordinate on V^*: $\langle \theta, \omega\rangle = \theta f$.

Since

$$X(\nu) = X(g\,\mathrm{d}^n x) = [X(g) + g \operatorname{div} X]\,\mathrm{d}^n x$$

and

$$[(X; \omega), (Y; \nu)] = ([X, Y]; X(\nu) - Y(\omega)),$$

the θ-terms of the matrix B are obtained as follows:

$$\begin{aligned}\langle \theta, X(\nu) - Y(\omega)\rangle &= \theta(X_i g_{,i} + gX_{i,i} - Y_i f_{,i} - fY_{i,i}) \\ &\equiv X_i(\theta\partial_i - \partial_i\theta)g - f(\theta\partial_i - \partial_i\theta)Y_i \\ &= X_i(-\theta_{,i})g + f(\theta_{,i})Y_i.\end{aligned} \tag{46}$$

Thus, the Λ^n piece of the Poisson bracket is

$$\{F, G\}_\theta = -\left(\frac{\delta G}{\delta M_i}\frac{\delta F}{\delta \theta} - \frac{\delta G}{\delta \theta}\frac{\delta F}{\delta M_i}\right)\theta_{,i}. \tag{47}$$

This part of the Poisson bracket will turn out to be summoned by elasticity theory in eq. (81) below.

Consider now $V = \Lambda^1$. Let $\omega = \omega_j\,\mathrm{d}x_j$, $\nu = \nu_j\,\mathrm{d}x_j \in V$ be typical elements in Λ^1 and let $\beta = (\beta_1, \ldots, \beta_n)$ denote coordinates on V^*: $\langle \beta, \omega\rangle = \beta_j \omega_j$. Since

$$X(\omega) = \left(X_i \frac{\partial}{\partial x_i}\right)(\omega_j\,\mathrm{d}x_j) = \left(X_i \omega_{j,i} + \omega_i \frac{\partial X_i}{\partial x_j}\right)\mathrm{d}x_j$$

and

$$[(X;\omega),(Y;\nu)] = ([X,Y]; X(\nu) - Y(\omega)),$$

we have (recall that $\langle M,[X,Y]\rangle$ terms are omitted)

$$\langle\beta, X(\nu) - Y(\omega)\rangle = \beta_j\left[X_i\nu_{j,i} - Y_i\omega_{j,i} + \nu_k\frac{\partial X_k}{\partial x_j} - \omega_k\frac{\partial Y_k}{\partial x_j}\right]$$
$$\equiv X_i(\beta_j\partial_i - \partial_k\beta_k\delta^i_j)\nu_j + \omega_i(\partial_j\beta_i - \beta_k\partial_k\delta^i_j)Y_j. \tag{48}$$

Thus

$$\{F,G\}_\beta = \frac{\delta G}{\delta M_i}(\beta_j\partial_i - \partial_k\beta_k\delta^i_j)\frac{\delta F}{\delta\beta_j} + \frac{\delta G}{\delta\beta_i}(\partial_j\beta_i - \beta_k\partial_k\delta^i_j)\frac{\delta F}{\delta M_j} \tag{49}$$

$$\equiv \beta_j\left(\frac{\delta G}{\delta M_i}\partial_i\frac{\delta F}{\delta\beta_j} - \frac{\delta F}{\delta M_i}\partial_i\frac{\delta G}{\delta\beta_j}\right) + \beta_k\left(\frac{\delta F}{\delta\beta_i}\partial_k\frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta\beta_i}\partial_k\frac{\delta F}{\delta M_i}\right). \tag{50}$$

Finally, we take $V = \Lambda^{n-1}$. Let $\omega = \omega_i\partial_i \lrcorner \mathrm{d}^n x$, $\nu = \nu_i\partial_i \lrcorner \mathrm{d}^n x$ be typical elements of Λ^{n-1}, where we denote $\partial_i = \partial/\partial x_i$ for convenience. Let $A = (A_1,\dots,A_n)$ be coordinates on V^*: $\langle A,\omega\rangle = A_i\omega_i$. Since

$$X(\nu) = (X_k\nu_{i,k} + \nu_i X_{k,k} - \nu_k X_{i,k})\partial_i \lrcorner \mathrm{d}^n x,$$

we have

$$\langle A, X(\nu) - Y(\omega)\rangle = A_i[X_k\nu_{i,k} + \nu_i X_{k,k} - \nu_k X_{i,k} - (Y_k\omega_{i,k} + \omega_i Y_{k,k} - \omega_k Y_{i,k})]$$
$$\equiv X_i(A_i\partial_j + A_{i,j} - A_{j,i})\nu_j + \omega_i(A_{i,j} - A_{j,i} + \partial_i A_j)Y_j, \tag{51}$$

which gives us the Poisson bracket piece

$$\{F,G\}_A = \frac{\delta G}{\delta M_i}(A_{i,j} - A_{j,i} + A_i\partial_j)\frac{\delta F}{\delta A_j} + \frac{\delta G}{\delta A_i}(A_{i,j} - A_{j,i} + \partial_i A_j)\frac{\delta F}{\delta M_j}. \tag{52}$$

Comparing this last piece with (10) we see that the sum

$$\{F,G\}_M + \{F,G\}_A + \{F,G\}_{\rho_1} + \{F,G\}_{\rho_2} \tag{53}$$

is exactly the Poisson bracket of MHD, which thus lives on the dual to the Lie algebra

$$\mathscr{D}(\mathbb{R}^n)\ominus(\Lambda^{n-1}(\mathbb{R}^n)\oplus C^\infty(\mathbb{R}^n)\oplus C^\infty(\mathbb{R}^n)). \tag{54}$$

Moreover, the Poisson bracket for the dual of the Lie algebra,

$$\mathscr{D}(\mathbb{R}^n)\ominus(\Lambda^{n-2}(\mathbb{R}^n)\oplus C^\infty(\mathbb{R}^n)\oplus C^\infty(\mathbb{R}^n)) \tag{55}$$

generates MHD equations (1)–(4) directly in terms of magnetic fields instead of vector potentials. Indeed, formula $B_{ij} = A_{i,j} - A_{j,i}$ is just the dual counterpart of the homomorphism of Lie algebras

$$\Delta = 1\ominus \mathrm{d}:\quad \mathscr{D}(\mathbb{R}^n)\ominus\Lambda^{n-2}\to\mathscr{D}(\mathbb{R}^n)\ominus\Lambda^{n-1}$$

provided one denotes dual coordinates on $(\Lambda^{n-2})^* = \Lambda^2$ as B_{ij}, and Hamiltonian H in (9) depends upon A_i only through B_{ij}.

We hasten to stress however that existence of Poisson brackets in both A- and B-space which, moreover, are compatible with respect to the "introduction-of-potentials map" $B = \mathrm{d}A$, is not to be expected. In general, the existence of one such bracket does not imply the existence of the other, and even if both brackets are present, they may be totally unrelated (as happens, e.g. in the theory of integrable systems). To add intrigue to surprise, we remark that in both of the remaining situations – multifluid plasmas and elasticity – there will again appear such compatible Hamiltonian structures in physical and potential spaces.

5. Multifluid plasma (MFP)

The multifluid plasma is a system of ideal, charged fluids which move together under self-consistent electromagnetic forces. The particle species are labeled by superscript s, with fluid velocities v_i^s; mass densities ρ^s; specific entropies η^s; self-consistent electric field E_i; and magnetic vector potential A_i. The vector potential produces magnetic field

$$B_{ij} = A_{i,j} - A_{j,i}.$$

The MFP equations consist of dynamical Maxwell equations for the self-consistent electromagnetic fields; conservation equations for mass and entropy of each species; and the MFP motion equation

$$\begin{aligned}
\dot{E}_i &= -B_{ij,j} - \sum_s a^s\rho^s v_i^s, \\
\dot{A}_i &= -E_i, \\
\dot{\rho}^s &= -(\rho^s v_j^s)_{,j}, \\
\dot{\eta}^s &= -v_j^s \eta^s_{,j}, \\
\dot{v}_i^s &= -v_j^s v_{i,j}^s - \frac{1}{\rho^s} p^s_{,i} + a^s(v_j^s B_{ji} + E_i).
\end{aligned} \tag{56}$$

(Note: in this section there is no summation convention on superscript s.) The static Maxwell source equation,

$$E_{i,i} = \sum_s a^s\rho^s \tag{57}$$

although nondynamical, is compatible with the flow, i.e. if initially true, eq. (57) will remain true under temporal evolution given by eqs. (56).

Hamilton's principle

$$\delta \int \mathrm{d}t\, \mathrm{d}^n x \mathscr{L} = 0 \tag{58}$$

implies the MFP equations (56) provided the Lagrangian density, $\mathscr{L}$, in the space of dependent variables $\{v_i^s, \rho^s, \eta^s, E_i, B_{ij}, A_i, \phi^s, \beta^s\}$ is given by

$$\begin{aligned}
\mathscr{L} = &\sum_s [\tfrac{1}{2}\rho^s(v^s)^2 - \rho^s e^s(\rho^s, \eta^s)] - \tfrac{1}{4}\operatorname{Tr} B^2 - \tfrac{1}{2}E^2 + A_i[\dot{E}_i + B_{ij,j} + \sum_s a^s\rho^s v_i^s] \\
&+ \sum_s \phi^s[\dot{\rho}^s + (\rho^s v_j^s)_{,j}] - \sum_s \beta^s[\dot{\eta}^s + v_j^s \eta^s_{,j}].
\end{aligned} \tag{59}$$

Here is how eqs. (56) are obtained. Variations with respect to $\{A_i, \phi^s, \beta^s\}$ impose respectively upon extremals of $\mathscr{L}$ the dynamical Maxwell source equation, conservation of mass for each species, and adiabatic convection as constraint conditions.

Variations in Hamilton's principle (58) with respect to the remaining variables $\{v_k^s, \rho^s, E_i, B_{ij}, \eta^s\}$ produce the following auxiliary equations:

$$\delta v_k^s: \quad \tilde{M}_k^s := \rho^s(v_k^s + a^s A_k) = \rho^s \phi_{,k}^s + \beta^s \eta_{,k}^s, \tag{60a}$$

$$\delta \rho^s: \quad \dot{\phi}^s + v_j^s \phi_{,j}^s = \tfrac{1}{2}(v^s)^2 - (e^s + p^s/\rho^s) + a^s A_j v_j^s, \tag{60b}$$

$$\delta E_i: \quad \dot{A}_i = -E_i, \tag{60c}$$

$$\delta B_{ij}: \quad B_{ij} = A_{i,j} - A_{j,i}, \tag{60d}$$

$$\delta \eta^s: \quad \dot{\beta}^s + (v_j^s \beta^s)_{,j} = \rho^s \partial e^s/\partial \eta^s, \tag{60e}$$

where $\partial e^s/\partial \eta^s = T^s$ is the species temperature.

The MFP motion equation then follows readily by algebraic manipulation of eqs. (60).

Eqs. (60b)–(60e) together with the constraint equations imposed by Lagrange multipliers $\{A_i, \phi^s, \beta^s\}$ can be quickly recast into canonical equations, with Hamiltonian density,

$$H = \sum_s [\tfrac{1}{2}\rho^s(v^s)^2 + \rho^s e^s(\rho^s, \eta^s)] - \tfrac{1}{4}\operatorname{Tr} B^2 + \tfrac{1}{2}E^2, \tag{61}$$

with canonical variables

$$\begin{aligned} &q_1^s = \varphi^s, \quad q_2^s = \eta^s, \quad q_i = E_i, \\ &p_1^s = \rho^s, \quad p_2^s = \beta^s, \quad p_i = A_i \end{aligned} \tag{62}$$

and with v_i^s in (61) taken from eq. (60a).

Moreover, the total momentum density for each particle species appears in the Clebsch representation as

$$\tilde{M}_i^s = \rho^s(v_i^s + a^s A_i) = p_\alpha^s q_{\alpha,i}^s$$

in terms of canonically conjugate variables.

We now describe the above computations in terms of maps between Poisson brackets. We begin with the canonical Poisson bracket

$$\{F, G\} = \sum_s \left(\frac{\delta F}{\delta p_\alpha^s}\frac{\delta G}{\delta q_\alpha^s} - \frac{\delta G}{\delta p_\alpha^s}\frac{\delta F}{\delta q_\alpha^s}\right) + \left(\frac{\delta F}{\delta E_i}\frac{\delta G}{\delta A_i} - \frac{\delta G}{\delta E_i}\frac{\delta F}{\delta A_i}\right). \tag{63}$$

At first, we leave $\boldsymbol{E}$ and $\boldsymbol{A}$ alone and consider maps from variables (p_α^s, q_α^s) into new variables $(\tilde{M}_i^s, \rho^s, \sigma^s)$:

$$\tilde{M}_i^s = p_\alpha^s q_{\alpha,i}^s; \qquad \rho^s = p_1^s; \qquad \sigma^s = p_1^s q_2^s. \tag{64}$$

For each s, these expressions are just eqs. (24) without A_i. As we know from the calculations for ideal hydrodynamics, the resulting bracket in $(\tilde{M}_i^s, \rho^s, \sigma^s)$ space is just

$$-\sum_s (\{F, G\}_{\tilde{M}^s} + \{F, G\}_{\rho^s} + \{F, G\}_{\sigma^s}), \tag{65}$$

which (up to the minus sign) is the bracket on the dual to the direct sum of Lie algebras,

$$\bigoplus_s [\mathscr{D}(\mathbb{R}^n) \ominus (C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n))]. \tag{66}$$

Taking into account the (A, E) part as well, we find ourselves in possession of the following bracket in $(\tilde{M}^s_i, \rho^s, \sigma^s, A_i, E_i)$ space:

$$-\{F, G\} = \sum_s \left\{\frac{\delta G}{\delta \rho^s} \partial_j \rho^s \frac{\partial F}{\delta M^s_j} + \frac{\delta G}{\delta \sigma^s} \partial_j \sigma^s \frac{\delta F}{\delta M_j} + \frac{\delta G}{\delta \tilde{M}^s_j}\left[\rho^s \partial_j \frac{\delta F}{\delta \rho^s} + \sigma^s \partial_j \frac{\delta F}{\delta \sigma^s} + (\tilde{M}^s_k \partial_j + \partial_j \tilde{M}^s_k) \frac{\delta F}{\delta \tilde{M}^s_k}\right]\right\} + \left(-\frac{\delta G}{\delta A_i}\frac{\delta F}{\delta E_i} + \frac{\delta F}{\delta A_i}\frac{\delta G}{\delta E_i}\right). \tag{67}$$

The next step is the invertible change of variables:

$$M^s_i = \tilde{M}^s_i - a^s \rho^s A_i = \rho^s v^s_i \qquad \text{(no sum on } s\text{)}. \tag{68}$$

All other variables remain the same. A new bracket is then easily produced from (67):

$$\begin{aligned}-\{F, G\} = \sum_s \Bigg\{&\left(\frac{\delta G}{\delta \rho^s} \partial_i \rho^s + \frac{\delta G}{\delta \sigma^s} \partial_i \sigma^s\right) \frac{\delta F}{\delta M^s_i} \\ &+ \frac{\delta G}{\delta M^s_k}\left[\rho^s \partial_k \frac{\delta F}{\delta \rho^s} + \sigma^s \partial_k \frac{\delta F}{\delta \sigma^s} + (M^s_i \partial_k + \partial_i M^s_k) \frac{\delta F}{\delta M^s_i} + a^s \rho^s \frac{\delta F}{\delta E_k} + a^s \rho^s (A_{i,k} - A_{k,i}) \frac{\delta F}{\delta M^s_i}\right] \\ &- \frac{\delta G}{\delta E_k} a^s \rho^s \frac{\delta F}{\delta M^s_k}\Bigg\} + \frac{\delta G}{\delta E_k}\frac{\delta F}{\delta A_k} - \frac{\delta G}{\delta A_k}\frac{\delta F}{\delta E_k}.\end{aligned} \tag{69}$$

This formula represents the Poisson bracket in the space of magnetic *potentials* plus other physical variables. But notice that A_i is involved only in the combination $A_{i,k} - A_{k,i} = B_{ik}$. Thus, one can immediately rewrite this bracket in the space of magnetic *fields* plus physical variables as

$$\begin{aligned}-\{F, G\} = \sum_s \Bigg\{&\left(\frac{\delta G}{\delta \rho^s} \partial_i \rho^s + \frac{\delta G}{\delta \sigma^s} \partial_i \sigma^s\right) \frac{\delta F}{\delta M^s_i} \\ &+ \frac{\delta G}{\delta M^s_k}\left[\rho^s \partial_k \frac{\delta F}{\delta \rho^s} + \sigma^s \partial_k \frac{\delta F}{\delta \sigma^s} + (M^s_i \partial_k + \partial_i M^s_k) \frac{\delta F}{\delta M^s_i} + a^s \rho^s \frac{\delta F}{\delta E_k} + a^s \rho^s B_{ik} \frac{\delta F}{\delta M^s_k}\right] \\ &- \frac{\delta G}{\delta E_k} a^s \rho^s \frac{\delta F}{\delta M^s_k}\Bigg\} + \frac{\delta G}{\delta B_{ij}}\left(\partial_i \frac{\delta F}{\delta E_j} - \partial_j \frac{\delta F}{\delta E_i}\right) + \frac{\delta G}{\delta E_k} \partial_i \frac{\delta F}{\delta B_{ik}}.\end{aligned} \tag{70}$$

In $\mathbb{R}^3$, this bracket is the same as the one reported by Spencer and Kaufman [7], obtained by another method.

6. Elasticity

The equations of ideal, nonlinear elasticity in Eulerian coordinates are [13, 14]

$$\dot{v}_i = -v_j v_{i,j} - \frac{1}{\rho} P_{ij,j}, \tag{71a}$$

$$\dot{\rho} = -(\rho v_j)_{,j}, \tag{71b}$$

$$\dot{\eta} = -v_j \eta_{,j}, \tag{71c}$$

$$\dot{X}^0_i = -v_j X^0_{i,j}. \tag{71d}$$

Here $X^0_i(x, t)$ denotes the Lagrangian coordinate: the initial location of the particle that occupies position x at time t. The Lagrangian coordinate X^0_i is a particle label; so it moves with the fluid and satisfies (71d). The tensor $X^0_{i,j} = F_{ij}(\boldsymbol{x}, t)$ is the displacement gradient, which measures the relative strain of the medium. Also $P_{ij}(x, t)$ is the stress tensor, ρ is the mass density, and η is the specific entropy. The stress tensor P_{ij} has the opposite sign of the "Cauchy stress," in order to agree with fluid dynamics convention.

In the elasticity equations, the stress tensor P_{ij} arises from the given equation of state for specific internal energy $e = e(F_{ij}, \eta)$ according to

$$P_{ij} = -\rho F_{ki} \frac{\partial e}{\partial F_{kj}}. \tag{72}$$

Thus, in terms of tensors P_{ij} and F_{ij} the first law of thermodynamics for elasticity may be expressed as

$$\mathrm{d}e = -\rho^{-1} F^{-1}_{ik} P_{ij}\, \mathrm{d}F_{kj} + T\, \mathrm{d}\eta, \tag{73}$$

where $T = \partial e/\partial \eta$ is the temperature, and the density $\rho(x, t)$ is related to the initial density $\rho_0 = \rho(x^0, 0)$ by

$$\rho = \rho_0 \det F$$

through the nonvanishing Jacobian, $\det F = |\partial(X^0_1, \ldots, X^0_n)/\partial(x_1, \ldots, x_n)|$.

The connection of the ideal elasticity equations with fluid dynamics is through the form of the dependence of $e(F_{ij}, \eta)$ on the displacement gradient. When the specific internal energy equation depends only upon $\det F$ and η, that is, $e = e(\det F, \eta)$ then motion equation (71a) reduces to the usual motion equation for ideal fluid dynamics and eq. (73) reduces to the First Law of thermodynamics for fluids, eq. (5).

A constrained Lagrangian density has been associated with ideal elasticity theory by Seliger and Whitham [14],

$$\mathscr{L} = \tfrac{1}{2}\rho v^2 - \rho e(F_{ij}, \eta) + \phi(\dot\rho + \operatorname{div} \rho \boldsymbol{v}) - \beta(\dot\eta + v \cdot \nabla\eta) - \gamma_k(\dot X^0_k + v \cdot \nabla X^0_k). \tag{74}$$

From Hamilton's principle for this Lagrangian density, the following variational equations result:

$$\delta v_k: \quad M_k =: \rho v_k = \rho\phi_{,k} + \beta\eta_{,k} + \gamma_j X^0_{j,k}, \tag{75a}$$

$$\delta\rho: \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{v^2}{2} - \left(e + \rho\frac{\partial e}{\partial\rho}\right), \tag{75b}$$

$$\delta\eta: \quad \dot\beta + (\beta v_k)_{,k} = \rho\frac{\partial e}{\partial\eta}, \tag{75c}$$

$$\delta X^0_k: \quad \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_k/\rho) = \frac{1}{\rho}\,\partial_j\left(\rho\frac{\partial e}{\partial F_{kj}}\right), \tag{75d}$$

$$\delta\varphi: \quad \dot\rho = -(\rho v_k)_{,k}, \tag{75e}$$

$$\delta\beta: \quad \dot\eta = -v_k\eta_{,k}, \tag{75f}$$

$$\delta\gamma_k: \quad \dot X^0_k = -v_l X^0_{k,l}. \tag{75g}$$

Seliger and Whitham [14] have shown that these equations reproduce the motion equation (71a) for ideal elasticity.

Moreover, the variational equations (75) are readily seen to be equivalent to canonical equations with canonical variables

$$p_\alpha \in (\rho, \beta, \gamma_j), \quad q_k \in (\phi, \eta, X^0_j), \tag{76}$$

and Hamiltonian density H given by

$$H = \tfrac{1}{2}\rho v^2 + \rho e(F_{ij}, \eta), \tag{77}$$

which is numerically equal to the energy density of the elastic medium. In terms of the canonical variables p^α, q^α the Clebsch representation (75a) for momentum density may again be expressed as $M_k = p^\alpha q^\alpha_{,k}$. Hamiltonian interpretation of these computations is easy now.

We start off with the canonical Poisson bracket in the Clebsch space:

$$\{F, G\} = \frac{\delta F}{\delta p_\alpha}\frac{\delta G}{\delta q_\alpha} - \frac{\delta G}{\delta p_\alpha}\frac{\delta F}{\delta q_\alpha}. \tag{78}$$

As in the MHD case, we introduce the variables

$$\rho = p_1, \qquad \sigma = p_1 q_2, \qquad M_i = p_\alpha q_{\alpha,i} \tag{79}$$

and in addition

$$X^0_i = q_{i+2}. \tag{80}$$

Restriction of canonical Poisson bracket (78) to the variables $\{\rho, \sigma, M_i, X^0_i\}$ readily produces the Poisson bracket for elasticity:

$$\{F, G\} = \{F, G\}_M + \{F, G\}_\rho + \{F, G\}_\sigma + \left(\frac{\delta G}{\delta M_j}\frac{\delta F}{\delta X^0_i} - \frac{\delta G}{\delta X^0_i}\frac{\delta F}{\delta M_j}\right) X^0_{i,j}. \tag{81}$$

The same bracket is obtained in [4] by a phenomenological method. Upon comparison of the last term in this bracket with formula (47) we see that the X^0_i part corresponds for each i, to the action of $\mathscr{D}(\mathbb{R}^n)$ on $\Lambda^n(\mathbb{R}^n)$. Thus, the Lie algebra responsible for this bracket is

$$L_{\mathrm{El}} = \mathscr{D}(\mathbb{R}^n) \ominus \left(C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n) \bigoplus_{i=1}^{n} \Lambda^n(\mathbb{R}^n)(i)\right), \tag{82}$$

and we can easily take advantage of this fact as follows.

Differential d: $\Lambda^{n-1} \to \Lambda^n$, as explained in section 4, induces the homomorphism

$$\mathrm{d}: \tilde{L}_{\mathrm{El}} \to L_{\mathrm{El}} \tag{83}$$

from the Lie algebra

$$\tilde{L}_{\mathrm{El}} = \mathscr{D}(\mathbb{R}^n) \ominus \left(C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n) \bigoplus_{i=1}^{n} \Lambda^{n-1}(\mathbb{R}^n)(i)\right) \tag{84}$$

into L_{El}. If we denote coordinates on the ith copy of $(\Lambda^{n-1})^*$ by $A^{[i]}_j$, $j = 1, \ldots, n$, as in (51), then the map dual to (83) will be given as

$$A^{[i]}_j = -X^0_{i,j} = -F_{ij}. \tag{85}$$

Now the Hamiltonian H depends upon X^0 only through F_{ij}, that is, through $A^{[i]}_j$. In other words, H in fact lives on the dual to $\tilde{L}_{\mathrm{El}}$, and it comes to elasticity under the pull back generated by (85). Thus, the Lagrange variables for elasticity, X^0_j, serve as "potentials" for dynamics on the dual to L_{El}.

Remark. The referee suggests that one might be able to reduce variables $\{F_{ij}\}$ to the smaller set of variables (Cauchy deformation tensor) $e_{ij} = (F^{t}F)_{ij} = F_{ki}F_{kj}$. This is indeed the case, which can be seen as follows. Let us concentrate only on the F_{ij} part of the bracket associated to $\tilde{L}_{\mathrm{El}}$, which can be written, using (52), as

$$\{J, I\}_F = \frac{\delta I}{\delta M_i}(\partial_k F_{si} - F_{sk,i})\frac{\delta J}{\delta F_{sk}} + \frac{\delta I}{\delta F_{si}}(F_{sk}\partial_i + F_{si,k})\frac{\delta J}{\delta M_k}, \tag{86}$$

where I and J are two functionals denoted earlier as F and G, and $\{F_{sj}\}_{j=1,\dots,n}$ are coordinates on the dual to the sth copy of $\Lambda^{n-1}(\mathbb{R}^n)$.

Suppose that I and J depend upon F_{ij} through e_{ij}. Since

$$\frac{\partial e_{\alpha\beta}}{\partial F_{ij}} = \delta_j^{\alpha}F_{i\beta} + \delta_j^{\beta}F_{i\alpha},$$

we get

$$\frac{\delta J}{\delta F_{ij}} = \frac{\partial e_{\alpha\beta}}{\partial F_{ij}}\frac{\delta J}{\delta e_{\alpha\beta}} = 2F_{i\beta}\frac{\delta J}{\delta e_{j\beta}} \tag{87}$$

which turns (86) into

$$\{J, I\}_e = \frac{\delta I}{\delta M_i}(2\partial_k e_{i\beta} - e_{k\beta,i})\frac{\delta J}{\delta e_{k\beta}} + \frac{\delta I}{\delta e_{i\beta}}(2e_{\beta k}\partial_i + e_{\beta i,k})\frac{\delta J}{\delta M_k}. \tag{88}$$

This piece of the bracket is linear in the variables e_{ij}. Therefore, we can find an appropriate Lie algebra. An easy computation shows that the total bracket

$$\{J, I\} = \{J, I\}_M + \{J, I\}_\rho + \{J, I\}_\sigma + \{J, I\}_e \tag{89}$$

comes from the Lie algebra

$$\mathscr{D}(\mathbb{R}^n) \ominus [C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n) \oplus (\mathscr{D}(\mathbb{R}^n) \otimes_s \Lambda^{n-1}(\mathbb{R}^n))], \tag{90}$$

where $\mathscr{D}(\mathbb{R}^n) \otimes_s \Lambda^{n-1}(\mathbb{R}^n)$ is a $C^\infty(\mathbb{R}^n)$-module generated by elements $\Sigma_i Y_i \otimes \omega_i$ such that $\Sigma_i Y_i \,\lrcorner\, \omega_i = 0$. The action of $\mathscr{D}(\mathbb{R}^n)$ on $\mathscr{D}(\mathbb{R}^n) \otimes_s \Lambda^{n-1}(\mathbb{R}^n)$, involved in (90), is given by

$$X(Y \otimes \omega) = [X, Y] \otimes \omega + Y \otimes X(\omega). \tag{91}$$

Variables e_{ij} are dual to $[\partial_i \otimes \partial_j \,\lrcorner\, \mathrm{d}^n x) + \partial_j \otimes (\partial_i \,\lrcorner\, \mathrm{d}^n x)]$. Notice that the module $\mathscr{D}(\mathbb{R}^n) \otimes_s \Lambda^{n-1}(\mathbb{R}^n)$ in (90) is isomorphic, as a $\mathscr{D}(\mathbb{R}^n)$-module, to $S^2(\mathscr{D}(\mathbb{R}^n) \otimes \Lambda^n(\mathbb{R}^n))$, which is the space of contravariant, symmetric, second-rank tensor densities, i.e. co-metric densities.

Note. Since this paper was submitted, some progress has been made concerning Hamiltonian structures of physical models in the presence of hydrodynamics. First, Poisson structures have been found both for Yang–Mills Vlasov plasmas and for fluids interacting self-consistently with Yang–Mills fields [18]; the new feature here is that the semi-direct product is formed (among other ingredients) by the Lie algebra $\mathscr{D}(\mathbb{R}^n)$ acting by *derivations* on the Lie algebra $C^\infty(\mathbb{R}^n) \otimes \mathfrak{g}$, $\mathfrak{g}$ a semisimple Lie algebra. In addition, Marsden et al. [17] have shown how to interpret Clebsch variables and Poisson brackets for systems of semi-direct product type, like the heavy top, MHD, and elasticity, from the point of view of reductions connected with appropriate Lie groups which are associated to Lie algebras of the present paper (except for the heavy top, which belongs to classical mechanics). Finally, Hamiltonian structures have been found for superfluid systems of ^{4}He and ^{3}He–A [19]. The new features here are: for ^{4}He, the Hamiltonian matrix has both a part which is linear in the variables and, thus, is associated to a Lie algebra L, say, and also a *constant* part, which corresponds to a generalized two co-cycle on L. For ^{3}He–A, multiply-knotted semi-direct products occur, formed by derivations of different Lie algebras. These are accompanied by a peculiar $(2 \dim \mathfrak{g} + 1)$-dimensional subalgebra, corresponding to the interaction between the density $\rho \in \Lambda^n(\mathbb{R}^n)$ and the order parameter $\psi \in C^\infty(\mathbb{R}^n, \mathfrak{g}) \otimes \mathbb{C}$.

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